

Def. $f(z) = \frac{P(z)}{Q(z)} \in \mathbb{C}(z)$, $P, Q \in \mathbb{C}[z]$, $f: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$
 $\leftarrow$ in lowest terms

$\perp \deg f := \max(\deg P, \deg Q)$, $\deg f \geq 2$

Fatou set $F(f)$ is the union of all open set
 on which $\{f^n\}$ is a normal family

$\perp$ Normal family $\mathcal{F}$: every $\{f_n\} \subset \mathcal{F}$ has a subsequence
 which converges loc. unif or diverges loc. unif.

Julia set $J(f) := \hat{\mathbb{C}} \setminus F(f)$

* $F(f)$ is open, $J(f)$ is closed

Prop. $F(f)$, $J(f)$ are fully invariant

$\perp$ forward invariant: $z \in A \Rightarrow f(z) \in A$

$\perp$ backward invariant: $z \in A, f(z') = z \Rightarrow z' \in A$

Proof. U open, $U = f^n(V)$, $\{f^n: U \rightarrow \hat{\mathbb{C}}\}_{n \geq 1}$ normal iff

$\{f^n: V \rightarrow \hat{\mathbb{C}}\}_{n \geq 1}$ is normal, so $F(f) \supset F(f^n)$
 $\Rightarrow J(f)$ is fully inv.

Prop. $F(f^n) = F(f)$, $J(f^n) = J(f)$

Proof. 1) $\{(f^n)^k\}_{k \geq 1} \subset \{f^n\}_{n \geq 1}$, so $F(f) \subset F(f^n)$.

2) $\{f^n\}_{n \geq 1} = \bigcup_{i=0}^{n-1} \{f^{kn+i}\}_{n \geq 1}$, any subsequence of

$\{f^n\}$ has a subsequence in one of the $\{f^{kn+i}\}$, norm. in
 eqn, so $F(f^n) \subset F(f)$.

Def. $z_0 \in \hat{\mathbb{C}}$ is periodic if $f^n(z_0) = z_0$. Period = minimal $n > 0$.

Multiplicator $\lambda = (f^n)'(z_0)$

$$= (\phi_u \circ f^n \circ \phi_u^{-1})'(w_0)$$

$$= (\phi_u \circ \phi_v^{-1})'(w_0) \cdot (\phi_v \circ f^n \circ \phi_v^{-1})'(w_0)$$

$$\cdot (\phi_v \circ \phi_u^{-1})'(w_0)$$

$$= (\phi_v \circ f^n \circ \phi_v^{-1})'(w_0)$$

$\lambda = f'(z_0) \cdot f'(f(z_0)) \cdots f'(f^{n-1}(z_0))$, and is the same for the whole orbit.

Periodic cycle is

- attracting: $|\lambda| < 1$, positively attracting $0 < |\lambda| < 1$,

superattracting if $\lambda = 0$ (cycle contains crit. pt.)

- repelling: $|\lambda| > 1$

- indifferent: $|\lambda| = 1$, parabolic $\lambda = e^{2\pi i p/q}$

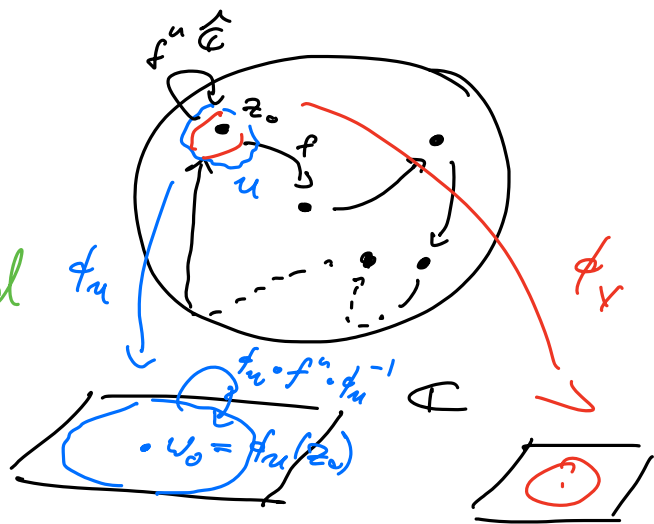
Ex: $f(z) = z^2$, $f'(z) = 2z$

Fixed pts:

0 $\lambda = 0$, superattracting $\swarrow$ crit. pts
1 $\lambda = 2$, repelling

$$\begin{array}{ccccc} \infty & \in & \hat{\mathbb{C}} \setminus \{0\} & \xrightarrow{z \mapsto z^2} & \hat{\mathbb{C}} \setminus \{0\} & \lambda = 0, \text{ superattracting} \\ \downarrow & \downarrow & \downarrow & & \downarrow & \\ \infty & \in & \mathbb{C} & \xrightarrow{w \mapsto w^2} & \mathbb{C} & \end{array}$$

$\frac{1}{2} = w$



$f^n(z) = z^{2^n}$, $(f^n)'(z) = 2^n z^{2^n-1}$, per. pts are roots of unity, $|\lambda| = 2^n$, always repelling

Ex: $f(z) = z^2 + \frac{1}{4}$

Fixed pts: $z_0 = \frac{1}{2}$, $\lambda = 1$ parabolic

Prop. Repelling periodic point lies in the Julia set.

Proof. $f(z_0) = z_0$, $|\lambda| > 1$. If $z_0 \in F(f)$, $\exists U \ni z_0$,

$\{f^{u_i}\}$ conv. loc. unif. U . But then $\{f^{u_i}\}^n$ conv. loc. unif.

However $(f^{u_i})'(z_0) = \lambda^{u_i} \rightarrow \infty$.

Def. $\mathcal{O}$ attracting orbit of period n

Attracting basin: $A(\mathcal{O}) = \{z \in \hat{\mathbb{C}} \mid f^{nk}(z) \rightarrow p \text{ as } k \rightarrow \infty \text{ for some } p \in \mathcal{O}\}$

Local coordinate

Prop. Every attracting basin lies in the Fatou set.

Proof. $f(0) = 0$, then a chart, $f(z) = \lambda z + a_2 z^2 + \dots$

$$|f(z) - \lambda z| \leq C|z|^2 \text{ on } \{|z| < r\} = B$$

Pick $|\lambda| < c < 1$, $0 < r < r_0$ s.t. $|\lambda| + Cr < s < 1$

Then on B ,

$$|f(z)| \leq \lambda|z| + C|z|^2$$

$$\Rightarrow |f^n(z_0)| < s^n r \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Loc. unif. conv. on $B \Rightarrow B \subset F(f)$, and

$$A(z_0) \subset \bigcup_{n \geq 0} f^{-n}(B) \subset F(f).$$

Ex: $f(z) = z^2$. $A(0) = \{ |z| < 1 \}$, $A(\infty) = \{ |z| > 1 \}$
 $\hat{F}(f)$ $\hat{F}(f)$

$\Rightarrow J(f) \subset \{ |z| = 1 \}$

$\{ z \mid z^{2^n} = 1 \} \subset J(f)$, $J(f)$ is closed, so

$J(f) = \{ |z| = 1 \}$.

Prop. Every parabolic periodic point belongs to the Julia set.

Proof. $f(0) = 0$, $f(z) = \lambda z + a_2 z^2 + \dots$ by continuity,

$f^n(z) = z + b_q z^q + \dots$, $q \geq 2$

$f^{n_k}(z) = z + k b_q z^q + \dots$, but if $\{ f^{n_k} \}_{k \in \mathbb{N}}$ converges

loc. unif. then we would $(f^{n_k})^{(q)}(0) = q! k b_q \rightarrow \infty$

so not conv in any neighborhood.

Ex: $f(z) = z^2 + \frac{1}{4}$, $z_0 = \frac{1}{2}$, $\lambda = 1$ Confluent

Prop. $J(f) \neq \emptyset$

Proof. If $\hat{F}(f) = \hat{\mathbb{C}}$, then $f^{n_i} \rightarrow g$ loc. unif. on $\hat{\mathbb{C}}$ to a rational $g \in \mathbb{C}(z)$. But $\deg(f^{n_i}) = (\deg f)^{n_i} \rightarrow \infty$.

Def. Grand orbit $GO(z_0, f) = \{ z \in \hat{\mathbb{C}} \mid f^u(z) = f^n(z_0) \}$
 for some $u, n \geq 0$

$z_0 \in \hat{\mathbb{C}}$ is exceptional for f if $GO(z_0, f)$ is finite. $E(f) = \{ \text{expt.} \}$

Describe by graphs
Prop. For any f , $|E(f)| \leq 2$, and $E(f)$ consist of superattracting point

Proof. 1) Suppose $|GO(z_0, f)| < \infty$. For each $z \in GO(z_0, f)$,
 pick a point $z' \in GO(z_0, f)$ s.t. $f(z') = z$ and $z' \neq z$.
 define an injection $GO(z_0, f) \rightarrow GO(z_0, f)$ by
 $z \mapsto z'$ and all are connected
 no fixed pts, $\forall$ so it is a cyclic permutation.
 $f: z_0 \mapsto z_1 \mapsto \dots \mapsto z_n = z_0$. But then they are
 the end. pts. $\rightarrow GO(z_0, f) \subset F(f)$, $\mathcal{E}(f) \subset F(f)$
 2) if $|\mathcal{E}(f)| \geq 3$, then $\hat{\mathbb{C}} \setminus \mathcal{E}(f)$ would be hyp,
 $\{f^n: \hat{\mathbb{C}} \setminus \mathcal{E}(f) \rightarrow \hat{\mathbb{C}} \setminus \mathcal{E}(f)\}$ would be normal,
 $\hat{\mathbb{C}} \setminus \mathcal{E}(f) \subset F(f)$, $J(f) = \emptyset$. Contradiction.
Thm. (Transitivity) $z_0 \in J(f)$. For all sufficiently
 small $U \ni z_0$ open, $V = \bigcup_{n \geq 0} f^n(U) = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, $V \supset J(f)$.
 Proof. if $|\hat{\mathbb{C}} \setminus V| \geq 3$, then V would be hyp,
 $\{f^n: V \rightarrow V\}$ would be normal, $V \subset F(f)$, but
 $z_0 \in U \subset V$ lies in $J(f)$.
 • if $z_1 \in \hat{\mathbb{C}} \setminus V$, then all preimages lie in $\hat{\mathbb{C}} \setminus V$,
 $\hat{\mathbb{C}} \setminus V$ is finite, so z_1 is periodic, hence $z_1 \in \mathcal{E}(f)$,
 $V \supset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, but if $U \subset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, then
 $V \subset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$.
Cor. if $J(f)$ has nonempty interior, then $J(f) = \hat{\mathbb{C}}$.
 Proof. if $U \subset J(f)$, then $V = \bigcup_{n \geq 0} f^n(U) \supset J(f)$
 and $\subset J(f)$, so $J(f) = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, but

$J(f)$ is closed, so $J(f) = \hat{J}$.

Cor. If $A = A(O)$ is a basin of attraction, then $\partial A = J(f)$.

Proof. 1) $z_0 \in J(f)$, $U \ni z_0$ open $\Rightarrow f^n(U) \cap A \neq \emptyset$
for some $n \Rightarrow U \cap A \neq \emptyset \Rightarrow J \subset \partial A$

2) $z_0 \in \partial A$, $U \ni z_0$ intersect both A and A^c .

Point is $A \rightarrow O$, while there is A^c orbit, so

there can be no convergent subseq. $U \ni z_0 \in J$, $\partial A \subset J$.